Lecture 16:

Galton - Watson Process

1°. The motivation comes from Galton's statistical investigation of the extinction of family names. This model is the simplest possible model for a population evolving in time. It's based on the assumption that the individuals give birth to a number of children independent of each other and all with the same distribution.

① We start the process with a singal individual which is the 0th generation of the population.

② This individual gives birth to a random number $X \in \mathbb{N}$ of children with $\mathbb{P}(X=0)>0$ and $\mu := \mathbb{E}[X] < \infty$. These are the 1st generation of the population.

③ For r-th individual in the n-th generation, it gives birth to children, the number of which,

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$X_{r,n}$, has the same distribution with X, and independent of all the other individuals in the generation.

Suppose we are given a doubly infinite sequence

$$\{X_{r,n} : r \in \mathbb{N}\setminus\{0\},\ n \in \mathbb{N}\}$$

of i.i.d. random variables, each with the same distribution with X:

$$\mathbb{P}(X_{r,n} = k) = \mathbb{P}(X = k).$$

Let Z_n denote the size of the n-th generation.

Then $\begin{cases} Z_{n+1} = X_{1,n} + X_{2,n} + \cdots + X_{Z_n,n}, & \forall\ n \geq 0. \\ Z_0 = 1. \end{cases}$

Q: Is $\{Z_n\}_{n\in\mathbb{N}}$ a Markov chain?

A: $\mathbb{P}(Z_{n+1} = z_{n+1} \mid (Z_i)_{i\in[0,n]} = (z_i)_{i\in[0,n]})$

$$= \mathbb{P}(Z_{n+1} = z_{n+1} \mid Z_n = z_n).$$

Q: What's the expected value of Z_n ?

A: By Tower Rule, $\forall n \in \mathbb{N}$,

$$
\begin{aligned}
&\mathbb{E}[Z_{n+1}] \\
&= \mathbb{E}[\mathbb{E}[Z_{n+1} \mid Z_n]] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[\sum_{r=1}^{Z_n} X_{r,n} \,\Big|\, Z_n\Big]\Big] \\
&= \sum_{j=1}^{\infty} \mathbb{E}\Big[\sum_{r=1}^{Z_n} X_{r,n} \,\Big|\, Z_n = j\Big] \cdot \mathbb{P}(Z_n = j) \\
&= \sum_{j=1}^{\infty} \mathbb{E}\Big[\sum_{r=1}^{j} X_{r,n}\Big] \cdot \mathbb{P}(Z_n = j) \\
&= \sum_{j=1}^{\infty} \Big(\sum_{r=1}^{j} \mathbb{E}[X_{r,n}]\Big) \cdot \mathbb{P}(Z_n = j) \\
&= \sum_{j=1}^{\infty} j \cdot \mu \cdot \mathbb{P}(Z_n = j) \\
&= \mu \cdot \mathbb{E}[Z_n].
\end{aligned}
$$

Thus, $\mathbb{E}[Z_n] = \mu^n \, \mathbb{E}[Z_0]$

$$
= \mu^n \xrightarrow{n \to \infty} \begin{cases} \infty, & \text{if } \mu > 1; \\ 1, & \text{if } \mu = 1; \\ 0, & \text{if } \mu < 1. \end{cases}
$$

Remark 16.1. We've essentially proved the Wald's Identity.

Theorem 16.1. (Wald's Identity)

Suppose $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of real-valued i.i.d. random variables, and N is a nonnegative integer-valued random variable that is independent of the sequence $\{X_n\}_{n\in\mathbb{N}}$. Suppose both have finite expectations, then $\mathbb{E}[X_1+X_2+\cdots+X_N]=\mathbb{E}[N]\,\mathbb{E}[X_1]$.

Q: How to compute $\mathbb{P}(\text{Survival}) = ?$

A: Notice that Survival means survival in each generation, i.e., $\forall n\in\mathbb{N}$, $Z_n\neq 0$. So,

$$\{\text{Survival}\} = \bigcap_{n=0}^{\infty}\{Z_n\neq 0\},$$ and

$$\{\text{Extinction}\} = \{\text{Survival}\}^c = \bigcup_{n=0}^{\infty}\{Z_n\neq 0\}^c = \bigcup_{n=0}^{\infty}\{Z_n=0\}.$$

Notice that $\{Z_n=0\}\subseteq\{Z_{n+1}=0\}$, $\forall n\in\mathbb{N}$.

Thus, $\mathbb{P}(\text{Extinction}) = \lim_{n\to\infty}\mathbb{P}(Z_n=0)$.

2°. Definition 16.1. (Generating Functions).

① We define the probability generating function G_X of a nonnegative integer-valued random variable X as the map $G_X : [0,1] \longrightarrow [0,1]$, where

$$G_X(\theta) := \mathbb{E}[\theta^X] = \sum_{k=0}^{\infty} \theta^k \mathbb{P}(X=k)$$

$$= \mathbb{P}(X=0) + \theta\,\mathbb{P}(X=1) + \theta^2 \mathbb{P}(X=2) + \cdots.$$

②. We define the moment generating function M_X of a nonnegative integer-valued random variable X as the map $M_X : (-r, r) \longrightarrow \mathbb{R}$, where

$$M_X(t) := \mathbb{E}[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} \mathbb{P}(X=k)$$

$$= 1 + t\,\mathbb{E}[X] + \frac{t^2}{2!}\mathbb{E}[X^2] + \cdots + \frac{t^n \mathbb{E}[X^n]}{n!} + \cdots$$

and $r > 0$ such that the expectation exists on $(-r, r)$.

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Remark 16.2. (Properties of the generating functions).

① $G_X(0) = \mathbb{P}(X=0)$, $G_X(1) = \sum_{k=0}^{\infty} \mathbb{P}(X=k) = 1$,

$G_X'(\theta) = \left(\sum_{k=0}^{\infty} \theta^k \mathbb{P}(X=k)\right)' = \sum_{k=0}^{\infty} (\theta^k \mathbb{P}(X=k))' = \sum_{k=1}^{\infty} k\cdot\theta^{k-1}\mathbb{P}(X=k)$

$= \mathbb{E}[X\cdot\theta^{X-1}]$.

In particular, $G_X'(1) = \mathbb{E}[X]$.

$$\frac{d^n}{d\theta^n}G_X(\theta) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!}\theta^{k-n}\mathbb{P}(X=k), \quad \forall n\in\mathbb{N}.$$

②. $M_X(0) = 1$.

$$\frac{d^n}{dt^n}M_X(t) = \sum_{k=0}^{\infty} k^n\cdot e^{tk}\,\mathbb{P}(X=k) = \mathbb{E}[X^n\cdot e^{tX}], \quad \forall n\in\mathbb{N}.$$

$$\frac{d^n}{dt^n}M_X(0) = \mathbb{E}[X^n], \quad \forall n\in\mathbb{N}.$$

Remark 16.3. To know the distribution of a random variable X, it is equivalent to find its probability generating function.

Pf. "$\Rightarrow$". Trivial from the definition of G_X.

"$\Leftarrow$". From the above formula, we know, $\forall n\in\mathbb{N}$,

$$\mathbb{P}(X=n) = \frac{1}{n!}\frac{d^n}{dz^n}G_X(0).$$ ∎

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3°. Let us come back to the Galton-Watson model.

Define $f_n(\theta) = \mathbb{E}[\theta^{Z_n}]$, $\forall n \geq 1$.

Then $f_1(\theta) = \mathbb{E}[\theta^{Z_1}] = \mathbb{E}[\theta^{X_{1,0}}] = \mathbb{E}[\theta^X]$, and

$$f_{n+1}(\theta) = \mathbb{E}[\theta^{Z_{n+1}}] = \mathbb{E}[\mathbb{E}[\theta^{Z_{n+1}} | Z_n]], \ \forall n \geq 1.$$

For each $k \in \mathbb{N}$, $\mathbb{E}[\theta^{Z_{n+1}} | Z_n = k]$

$$= \mathbb{E}[\theta^{X_{1,n} + \cdots + X_{k,n}} | Z_n = k]$$

$$= \mathbb{E}[\theta^{X_{1,n}} \cdot \cdots \cdot \theta^{X_{k,n}} | Z_n = k]$$

$$= \mathbb{E}[\theta^{X_{1,n}} \cdot \cdots \cdot \theta^{X_{k,n}}]$$

why?

$$= \mathbb{E}[\theta^{X_{1,n}}] \cdot \cdots \cdot \mathbb{E}[\theta^{X_{k,n}}]$$

$$= [f_1(\theta)]^k.$$

Thus,

$$f_{n+1}(\theta) = \mathbb{E}[\mathbb{E}[\theta^{Z_{n+1}} | Z_n]] = \mathbb{E}[[f_1(\theta)]^{Z_n}] = f_n(f_1(\theta)), \ \forall n \geq 1.$$

By induction, $f_{n+1}(\theta) = \underbrace{f_1 \circ f_1 \circ \cdots \circ f_1}_{\text{"}n+1\text{" times}}(\theta) = f_1(f_n(\theta))$, $\forall n \geq 1$.

Assume that $s = \lim_{n\to\infty} f_n(0)$ exists.

Since f_1 is continuous, taking limits at both sides gives

$$s = \lim_{n\to\infty} f_{n+1}(0) = f_1\left(\lim_{n\to\infty} f_n(0)\right) = f_1(s).$$

That is, s is a fixed point of the function f_1.

On the other hand, $f_n(0) = \mathbb{P}(Z_n = 0)$. Therefore,

$$\mathbb{P}(\text{Extinction}) = \lim_{n\to\infty} \mathbb{P}(Z_n = 0) = \lim_{n\to\infty} f_n(0) = s.$$

This is to say, $\mathbb{P}(\text{Extinction})$ is a fixed point of f_1.

Notice

$$(f_1)'(\theta) = \sum_{k=1}^{\infty} k \cdot \theta^{k-1} \mathbb{P}(X=k) \geq 0, \quad \forall\, \theta \in [0,1].$$

and

$$(f_1)''(\theta) = \sum_{k=2}^{\infty} k(k-1) \cdot \theta^{k-2} \mathbb{P}(X=k) \geq 0, \quad \forall\, \theta \in [0,1].$$

Thus, f_1 is increasing and convex on $[0,1]$.

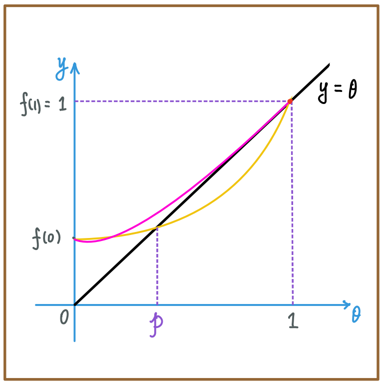
Combining Remark 16.2 and our assumptions on μ and $\mathbb{P}(X=0)$, one has $f_1(0) = \mathbb{P}(X=0) > 0$, $f_1(1) = 1$, and $f'(1) = \mathbb{E}[X] = \mu < \infty$.

why? Therefore, there are at most 2 fixed points.

Case I: $\mu = f'(1) \le 1$.

In this case, $f(1) = 1$ is the unique fixed point on $[0,1]$.

Then $\mathbb{P}[\text{Extinction}] = 1$.



Case II: $\mu = f'(1) > 1$.

In this case, there will be two fixed points on $[0,1]$, say p and 1, where $0 < p < 1$.

Since f is increasing and $p > 0$, $p = f(p) \ge f(0)$.

Moreover, $p = f_n(p) \ge f_n(0)$, $\forall n \ge 1$.

Taking limits at both sides yields

$$p \ge \lim_{n\to\infty} f_n(0) = \mathbb{P}(\text{Extinction}).$$

Because $\mathbb{P}(\text{Extinction})$ is a fixed point of f, that does not exceed p, we have

$$\mathbb{P}(\text{Extinction}) = p < 1.$$

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Theorem 16.2. If $\mathbb{E}[X] > 1$, then the extinction probability is the unique root of the equation $p = f_1(p)$ that lies in $(0,1)$.

If $\mathbb{E}[X] \leq 1$, then the extinction probability is 1.

This is the end of this lecture!